

ON SUPERSTABLE EXPANSIONS OF FREE ABELIAN GROUPS

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ABSTRACT. We prove that $(\mathbb{Z}, +, 0)$ has no proper superstable expansions of finite Lascar rank. Nevertheless, this structure equipped with a predicate defining powers of a given natural number is superstable of Lascar rank ω . Additionally, our methods yield other superstable expansions such as $(\mathbb{Z}, +, 0)$ equipped with the set of factorial elements.

1. INTRODUCTION

This paper fits into the general framework of adding a new predicate to a well behaved structure and asking whether the obtained structure is still well behaved. A similar line of thought is to impose the desired properties on the expanded structure and ask for which predicates these properties are fulfilled. Even more, one might ask whether there exist proper expansions fulfilling the desired properties.

Many results that belong to the above mentioned framework have been obtained by various authors. For example Pillay and Steinhorn proved that there are no (proper) o-minimal expansions of (\mathbb{N}, \leq) . On the other hand, Marker [3] proved that there are (proper) strongly minimal expansions of (\mathbb{N}, s) , i.e. the natural numbers with the successor function. In a more abstract context Baldwin and Benedikt proved that if \mathcal{M} is a stable structure and I is a *small* set of indiscernibles then (\mathcal{M}, I) is still stable. Finally, Chernikov and Simon [2] proved the analogous result for NIP theories, i.e. NIP is preserved after naming a *small* indiscernible sequence.

In this short paper we are mainly interested in (finitely generated) free abelian groups. We are motivated by the recent addition of torsion-free hyperbolic groups to the family of stable groups (see [6]). In a torsion-free hyperbolic group centralizers of (non-trivial) elements are infinite cyclic and one is interested in the induced structure on them. It seems that understanding the induced structure on these centralizers boils down to understanding whether they are superstable and if so calculate their Lascar rank.

Our main result generalizes a theorem in the thesis of the second named author proving that every Lascar rank 1 expansion of $(\mathbb{Z}, +, 0)$ is a pure group (see [7, Theorem 8.2.3]).

Theorem 1. *There are no (proper) superstable finite Lascar rank expansions of $(\mathbb{Z}, +, 0)$.*

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We also show that one cannot strengthen the above result any further by proving:

Theorem 2. *The theory of $(\mathbb{Z}, +, 0, \Pi_q)$ is superstable of Lascar rank ω , where Π_q denotes the set of powers of a natural number q .*

In fact, our methods can be used to provide other superstable expansions by adding other sets such as sets of the form $\{q^{p^n}\}_{n < \omega}$ for some natural numbers p, q or the set of factorial elements, see Proposition 4.2. On the other hand, if one moves to higher rank free abelian groups Theorem 1 is no longer true, and it is not hard to find proper superstable Lascar rank 1 expansions of $(\mathbb{Z}^n, +, 0)$, for $n \geq 2$. The main reason being that there exist finite index subgroups of \mathbb{Z}^n (for $n \geq 2$) that are not definable in $(\mathbb{Z}^n, +, 0)$. Still, we record, that a superstable finite Lascar rank expansion of $(\mathbb{Z}^n, +, 0)$ is one-based and has Lascar rank at most n .

While checking our results, the second author figured out in a talk of Bruno Poizat that Theorem 2 was already proved in [5, Théorème 25]. Nevertheless, as both approaches are completely distinct we believe that it is worthy recording our result since, as we have already pointed out, it yields distinct examples. Moreover, to our knowledge, Theorem 1 was unknown. The essential tools to prove it come from geometric stability. We combine results from Hrushovski's thesis together with Buechler's dichotomy theorem, the characterization of one-based groups by Hrushovski-Pillay and a result on one-based types due to Wagner.

2. FINITE RANK EXPANSIONS

The aim of this section is to study superstable expansions of finite Lascar rank of the structure $(\mathbb{Z}^n, +, 0)$. We assume the reader is familiarized with the general theory of geometric stability, see [4, 8] as a reference. In addition we require the following result which characterizes subgroups of finitely generated free abelian groups.

Fact 2.1. *Let G be a subgroup of \mathbb{Z}^n . Then there is a basis (z_1, \dots, z_n) of \mathbb{Z}^n and a sequence of natural numbers d_1, \dots, d_k (with d_i dividing d_{i+1} for $i < k$), such that $(d_1 z_1, \dots, d_k z_k)$ forms a basis of G .*

One can use Fact 2.1 to prove the following lemma, which we consider as being part of the folklore.

Lemma 2.2. *Let G be a subgroup of \mathbb{Z}^n . Then G is definable in $(\mathbb{Z}, +, 0)$.*

Now, we prove Theorem 1.

Proof of Theorem 1. Consider a finite Lascar rank expansion $\mathcal{Z} = (\mathbb{Z}, +, 0, \dots)$ of $(\mathbb{Z}, +, 0)$, and let $\Gamma \succeq \mathcal{Z}$ be an enough saturated elementary extension. As Γ has finite Lascar rank, its principal generic type is non-orthogonal to a type q of Lascar rank one and hence, we can find an \emptyset -definable normal subgroup H of infinite index in Γ in a way that Γ/H is \mathcal{Q} -internal, where \mathcal{Q} is the family of all \emptyset -conjugates of q . In fact, since H is defined without parameters, the subgroup $H \cap \mathbb{Z}$ has infinite index in \mathbb{Z} , hence $H \cap \mathbb{Z}$ must be trivial, and so is H . This yields that Γ is \mathcal{Q} -internal. On the other hand, as Γ is not ω -stable, by Buechler's dichotomy theorem q must be a one-based type and so are all its conjugates. Thus Γ is one-based by [9, Corollary 12], and so is the theory of \mathcal{Z} . Thus, by the characterization of one-based stable groups [4, Corollary 4.4.8], every definable subset of \mathbb{Z}^n in the expanded structure is a boolean combination of cosets of definable subgroups of \mathbb{Z}^n and therefore, any

definable set in the theory of \mathcal{Z} is already definable in the theory of $(\mathbb{Z}, +, 0)$ by the previous lemma, as desired. \square

We note, in contrast, that not all subgroups of \mathbb{Z}^n are definable in $(\mathbb{Z}^n, +, 0)$. For example, the finite index subgroup $3\mathbb{Z} \oplus 2\mathbb{Z}$ of \mathbb{Z}^2 is not definable in $(\mathbb{Z}^2, +, 0)$, and of course any non-trivial infinite index subgroup of \mathbb{Z}^n , for $n \geq 2$, is not definable in $(\mathbb{Z}^n, +, 0)$.

Theorem 2.3. *Any finite Lascar rank expansion of $(\mathbb{Z}^n, +, 0)$ is one-based and has Lascar rank at most n .*

Proof. Consider a finite Lascar rank expansion $\mathcal{Z} = (\mathbb{Z}^n, +, 0, \dots)$ of $(\mathbb{Z}^n, +, 0)$. A similar argument as in the previous theorem yields that the theory of \mathcal{Z} is one-based. For this, let $\Gamma \succeq \mathcal{Z}$ be an enough saturated model. As it has finite Lascar rank by assumption, the general theory yields the existence of a finite series of \emptyset -definable normal subgroups

$$\Gamma = H_0 \supseteq H_1 \supseteq \dots \supseteq H_{m+1} \supseteq \{0\}$$

such that H_n is finite and each factor H_i/H_{i+1} is infinite and internal to a family \mathcal{Q}_i of \emptyset -conjugates of some type q_i of Lascar rank one. Since free abelian groups are torsion-free they do not have any finite (non-trivial) subgroups, and so neither does Γ . This implies that H_{m+1} is trivial. Furthermore, by Fact 2.1 we obtain that no infinite quotient of \mathbb{Z}^n is ω -stable. As all subgroups H_i are \emptyset -definable, we deduce that the quotients H_i/H_{i+1} cannot have ordinal Morley rank, and neither do the types from the families \mathcal{Q}_i . Whence, we conclude by Buechler's dichotomy theorem that all of them are one-based, and so is Γ again by [9, Corollary 12].

To see that the expansion \mathcal{Z} has Lascar rank at most n , consider the structure $\mathcal{Z}_{\text{proj}}$ given as $(\mathbb{Z}^n, +, 0, P_1, \dots, P_n)$, where the predicate P_i is interpreted as the projection of \mathbb{Z}^n onto its i th coordinate. It is clear that $\mathcal{Z}_{\text{proj}}$ is interpretable in $(\mathbb{Z}, +, 0)$ and so it has Lascar rank n . On the other hand, since \mathcal{Z} is one-based, it is interpretable in $\mathcal{Z}_{\text{proj}}$ by the characterization of one-based stable groups [4, Corollary 4.4.8] and thus, it has Lascar rank at most n . \square

Remark 2.4. Observe that the proof yields that any superstable finite Lascar rank expansion of $(\mathbb{Z}^n, +, 0)$ is interpretable in the structure $\mathcal{Z}_{\text{proj}}$.

3. SUPERSTABLE EXPANSIONS OF $(\mathbb{Z}, +, 0)$

In this section we shall see that there are proper superstable expansions of $(\mathbb{Z}, +, 0)$, necessarily, by Theorem 1, of infinite Lascar rank.

Definition 3.1. Let \mathcal{L} be a first-order language and $P(x)$ a unary predicate. We denote by \mathcal{L}_P the first-order language $\mathcal{L} \cup \{P\}$. We say that an \mathcal{L}_P -formula $\phi(\bar{y})$ is bounded (with respect to P) if it has the form $Q_1 x_1 \in P \dots Q_n x_n \in P \psi(\bar{x}, \bar{y})$, where the Q_i 's are quantifiers and $\psi(\bar{x}, \bar{y})$ is an \mathcal{L} -formula.

The following theorem will be useful for proving Theorem 2, we refer the reader to [1] for the proof.

Theorem 3.2. *Let \mathcal{M} be an \mathcal{L} -structure and $A \subseteq M$. Consider (\mathcal{M}, A) as a structure in the expanded language $\mathcal{L}_P := \mathcal{L} \cup \{P\}$. Suppose every \mathcal{L}_P -formula in (\mathcal{M}, A) is equivalent to a bounded one. Then, for every $\lambda \geq |\mathcal{L}|$, if both \mathcal{M} and A_{ind} are λ -stable, then (\mathcal{M}, A) is λ -stable.*

Let \equiv_n be the congruence modulo n relation on the integers. Observe that $a \not\equiv_n b$ is equivalent to $a \equiv_n b + 1 \vee a \equiv_n b + 2 \vee \dots \vee a \equiv_n b + (n - 1)$, and hence we get the following remark.

Remark 3.3. Let \mathcal{L}_{mod} be the language of groups expanded with countably many 2-place predicates. We recall that an \mathcal{L}_{mod} -formula $\phi(\bar{x})$, is equivalent, in $(\mathbb{Z}, +, 0, \{\equiv_n\}_{n < \omega})$, to a finite disjunction of formulas of the form:

$$\begin{array}{ccccccc} t_1(\bar{x}) = 0 & \wedge & \dots & \wedge & t_k(\bar{x}) = 0 \\ r_1(\bar{x}) \neq 0 & \wedge & \dots & \wedge & r_l(\bar{x}) \neq 0 \\ s_1(\bar{x}) \equiv_{n_1} 0 & \wedge & \dots & \wedge & s_m(\bar{x}) \equiv_{n_m} 0 \end{array}$$

where $t_i(\bar{x}), s_i(\bar{x}), r_i(\bar{x})$ are terms in the above language.

Before moving to the next lemma we introduce for convenience the notion of “consecutive elements” of a subset of the integers. We say that two distinct elements a_1, a_2 of $\Pi \subseteq \mathbb{Z}$ with $a_1 < a_2$ are *consecutive in Π* , if there is no $a \in \Pi$ such that $a_1 < a < a_2$.

Since our main focus will be on the subset of the integers consisting of powers of some natural number, we fix the following notation $\Pi_q := \{q^n \mid 1 \leq n < \omega\}$ for some natural number q .

Lemma 3.4. *Let q be a natural number. Let \bar{b} be a tuple in \mathbb{Z} and $\phi(\bar{b}, y, \alpha)$ be an \mathcal{L} -formula, where \mathcal{L} is the language of groups. Suppose $\Gamma(y) := \{\phi(\bar{b}, y, \alpha) \mid \alpha \in \Pi_q\}$ is consistent with $\text{Th}(\mathbb{Z}, +, 0)$. Then there exists $c \in \mathbb{Z}$ realizing the set $\Gamma(y)$.*

Proof. We may assume that $\phi(\bar{x}, y, \alpha)$ is a formula as in Remark 3.3. If we fix some α_0 in Π_q , then each disjunctive clause in $\phi(\bar{b}, y, \alpha_0)$ asserts that y is equal to some element from a finite list of elements in \mathbb{Z} , and y is not equal to any element from a finite list of elements in \mathbb{Z} and y belongs to the intersection of finitely many cosets of fixed subgroups of \mathbb{Z} , where these fixed subgroups only depend on ϕ (not \bar{b} or α_0).

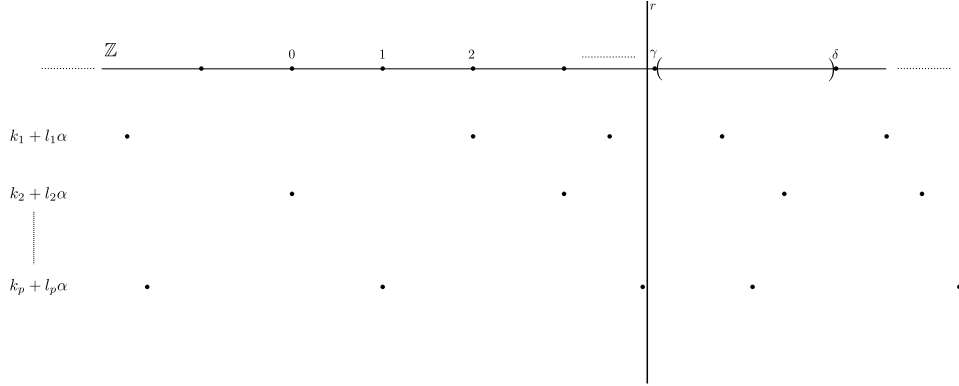
Our assumption that $\Gamma(y)$ is consistent implies that for each α_0 in Π_q we may choose a disjunctive clause in $\phi(\bar{b}, y, \alpha_0)$ such that the set of these clauses is again consistent. Note that if one of the chosen clauses involves an equality, then the result holds trivially. So we will assume that no equality is involved in any disjunctive clause of ϕ . On the other hand the intersection of cosets of subgroups of a group is either empty or a coset of the intersection of the subgroups, thus we may assume that a disjunctive clause that involves congruence modulo relations, it involves exactly one.

Next we prove that a finite union of sets of the form $\{k + l \cdot \alpha \mid \alpha \in \Pi_q\}$ cannot cover any coset of any (non-trivial) subgroup of \mathbb{Z} . Suppose not, and let

$$n\mathbb{Z} + m \subseteq \{k_1 + l_1 \cdot \alpha \mid \alpha \in \Pi_q\} \cup \{k_2 + l_2 \cdot \alpha \mid \alpha \in \Pi_q\} \cup \dots \cup \{k_p + l_p \cdot \alpha \mid \alpha \in \Pi_q\}.$$

We may assume that there are l_i 's which are positive. Then we can find a natural number $r > 0$ which is bigger than any k_i with $l_i < 0$, and so that any two consecutive elements bigger than r in any set of the above union differ by distance more than $(p+1) \cdot n$. Thus, we get an interval (γ, δ) in \mathbb{Z} , that contains $p+1$ elements of $n\mathbb{Z} + m$ but not more than p elements of the above union, a contradiction.

Now the consistency of $\Gamma(y)$ implies that y belongs to the intersection of finitely many cosets of subgroups of \mathbb{Z} and y is not equal to any element of a union of sets of the form $\{k + l \cdot \alpha \mid \alpha \in \Pi_q\}$. By our previous claim, a solution can be found in \mathbb{Z} and this finishes the proof. \square

FIGURE 1. The union of finitely many sets of the form $\{k + l\alpha \mid \alpha \in \Pi_q\}$.

Remark 3.5. We note that the above proof is valid for any subset of the natural numbers $A := \{a_i \mid i < \omega\}$, for which the value $a_{i+1} - a_i$ is strictly increasing for all $i > k$ for a given natural number k .

Now we are able to prove the following technical lemma.

Lemma 3.6. *Let q be a natural number. Let \mathcal{L} be the language of groups and $P(x)$ be a unary predicate. Let $\mathcal{Z} := (\mathbb{Z}, +, 0, \Pi_q)$ be an \mathcal{L}_P -structure.*

Let $\phi(\bar{x}, y, \alpha)$ be an \mathcal{L} -formula. Then there exists $k < \omega$ such that:

$$\mathcal{Z} \models \forall \bar{x} ((\forall \alpha_0 \in P \dots \forall \alpha_k \in P \exists y \phi(\bar{x}, y, \alpha_0) \wedge \phi(\bar{x}, y, \alpha_1)) \rightarrow \exists y \forall \alpha \in P \phi(\bar{x}, y, \alpha)).$$

Proof. Since $(\mathbb{Z}, +, 0)$ has nfcf we can assign to each formula ϕ a natural number k such that any set of instances of the formula ϕ is consistent if and only if it is k -consistent. By Lemma 3.4 if a set $\{\phi(\bar{b}, y, \alpha) \mid \alpha \in \Pi_q\}$ is consistent, then a solution can be found in \mathbb{Z} and this is enough to conclude. \square

The following proposition is an easy corollary of Lemma 3.6 and the proof is left to the reader, see [1, Proposition 2.1].

Proposition 3.7. *Let q be a natural number. Let \mathcal{L} be the language of groups and $P(x)$ be a unary predicate. Let $\mathcal{Z} := (\mathbb{Z}, +, 0, \Pi_q)$ be an \mathcal{L}_P -structure. Then every \mathcal{L}_P -formula in \mathcal{Z} is bounded.*

As a consequence we deduce:

Corollary 3.8. *Let q be a natural number. Let \mathcal{L} be the language of groups and $P(x)$ be a unary predicate, and let $(\Gamma, +', 0, \Pi'_q) \equiv (\mathbb{Z}, +, 0, \Pi_q)$ be \mathcal{L}_P -structures. Two tuples of Γ realize the same \mathcal{L}_P -formulas over any set of parameters $C \subseteq \Gamma$ whenever they realize the same \mathcal{L} -formulas over $\Pi'_q \cup C$.*

Proof. Let a and b be two tuples realizing the same \mathcal{L} -formulas over Π'_q, C . It is easy to see by induction on the number of quantifiers that a and b realize the same formulas of the form

$$Q_1 x_1 \in P \dots Q_n x_n \in P \psi(\bar{x}, \bar{y}),$$

where the Q_i 's are quantifiers and $\psi(\bar{x}, \bar{y})$ is an $\mathcal{L}(\Pi'_q \cup C)$ -formula. Hence, we conclude by Proposition 3.7. \square

Our last task is to prove that the induced structure on the subset of the integers that consists of powers of some natural number, coming from $(\mathbb{Z}, +, 0)$, is tame. Recall that if B is a subset of the domain M , of a first order structure \mathcal{M} , then by the *induced structure on B* we mean the structure with domain B and predicates for every subset of B^n of the form $B^n \cap \phi(M^n)$, where $\phi(x)$ is a first order formula (over the empty set). We denote this structure by B^{ind} .

Proposition 3.9. *Let q be a natural number. The structure Π_q^{ind} (with respect to $(\mathbb{Z}, +, 0)$) is superstable and has Lascar rank one.*

The proof is split in a series of lemmata. We first prove some results, we believe well known, in the spirit of Diophantine analysis.

Lemma 3.10. *Let q be some natural number. Let $k < n$ be natural numbers such that n is co-prime with q , and let $[k]_n$ denote the congruence class of k modulo n . Then $\Pi_q \cap [k]_n = \{q^{m_0 + \varphi(n) \cdot m} : m < \omega\}$, where $\varphi(n)$ is the Euler's phi function and m_0 is the smallest natural number for which $q^{m_0} \equiv k \pmod{n}$.*

Proof. We first note that if k, n are not co-prime then the intersection of $[k]_n$ with Π_q is empty. The common factor of k and n does not contain a factor of q since n is co-prime with q , and it should appear as factor in any element of $k + n \cdot \mathbb{Z}$.

We now assume that k, n are co-prime and we fix k, n, m_0 satisfying the hypothesis of the lemma. We define λ_m recursively as follows:

$$\begin{aligned} \lambda_0 &:= \frac{q^{m_0} - k}{n} \\ \lambda_{m+1} &:= \lambda_m \cdot b^{\varphi(n)} + k \cdot \frac{q^{\varphi(n)} - 1}{n}, \text{ for } 0 \leq m < \omega. \end{aligned}$$

Note that, by Euler's theorem, all the λ_m 's are integers. Furthermore, one can easily see, by induction on m , that $\lambda_m \cdot n + k$ is a power of q of the form $q^{m_0 + \varphi(n) \cdot m}$ and therefore $\{\lambda_m \cdot n + k \mid m < \omega\} \subseteq \Pi_q \cap [k]_n$.

In fact, the other inclusion also holds. To see this, let q^l be an arbitrary power of q . We may assume that $l > m_0$, since m_0 is the smallest natural number satisfying the hypothesis. Then we can find some m such that $l = m_0 + \varphi(n) \cdot m + s$ with $s < \varphi(n)$. As $\varphi(n)$ is the order of the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^\times$, we get $q^s \in [1]_n$ only when $s = 0$. Since k, n are co-prime k has a multiplicative inverse modulo n . Therefore

$$q^l = q^{m_0 + \varphi(n) \cdot m} \cdot q^s \equiv_n k \cdot q^s \equiv_n k \text{ if and only if } s = 0,$$

and this concludes the proof. \square

Remark 3.11. Let q be some natural number. Assume n is a power of a prime which is not co-prime with q , then the intersection of Π_q with $[k]_n$ is either finite or co-finite in Π_q .

If $\bar{a} := (q^{k_1}, \dots, q^{k_n})$ is an element of Π_q^n , then we denote by $\langle \bar{a} \rangle^+$ the following set of positive powers of \bar{a} , $\{(q^{m \cdot k_1}, \dots, q^{m \cdot k_n}) \mid m < \omega\}$.

Lemma 3.12. *Let $k_1 x_1 + \dots + k_n x_n = 0$ be an equation over the integers and $S \subseteq \mathbb{Z}^n$ be its solution set. Then $S \cap \Pi_q^n$ is either empty or the union of finitely many sets of positive powers of elements in Π_q^n .*

Proof. It is immediate that if a tuple (a_1, \dots, a_n) of Π_q^n is a solution of

$$k_1 x_1 + \dots + k_n x_n = 0,$$

then each element of $\langle (a_1, \dots, a_n) \rangle^+$ is a solution as well. Consider the set of all such (maximal for inclusion) sets of positive powers of solutions of the above equation. Suppose, for a contradiction, that this set is infinite. Then by an easy pigeonhole principle argument there exists a co-ordinate $i \leq n$ such that the i -th element of the “generating” tuple is arbitrarily larger than the rest. Thus, we can choose a_i to be more than $|k_1| \cdot |k_2| \cdot \dots \cdot |k_{i-1}| \cdot |k_{i+1}| \cdot \dots \cdot |k_n|$ times larger from each a_j with $j \leq n$ and $j \neq i$. Now it is easy to see that $k_1 a_1 + \dots + k_n a_n \neq 0$, a contradiction. \square

Lemma 3.13. *Let q be some natural number. Let $\mathcal{N} := (\mathbb{N}, s, \{Q_{k,n}\}_{n < \omega, k < n})$ be a first order structure where the function symbol s is interpreted as the successor function and the predicate $Q_{k,n}$ is interpreted as the set of natural numbers which are residual to k modulo n . Then Π_q^{ind} is definably interpreted in \mathcal{N} .*

Proof. Through out the proof the symbol s^m will be used to denote $s \circ s \circ \dots \circ s$ m -times. We also allow m to be negative, in which case s^m denotes the composition of the predecessor function m -times (which is clearly definable).

We first interpret Π_q to be the domain of \mathcal{N} . Now let P be a predicate of Π_q^{ind} . By the construction of Π_q^{ind} we have that P is a subset of the form $\phi(\mathbb{Z}^n) \cap \Pi_q^n$ for some quantifier free formula ϕ in $(\mathbb{Z}, +, 0, \{\equiv_n\}_{n < \omega})$. Since a quantifier free formula is a boolean combination of formulas of the form $t(\bar{x}) = 0$ and $s(\bar{x}) \equiv_l 0$, we only need to interpret in \mathcal{N} solution sets of equations and congruence relations of the above simple form intersected with Π_q^n .

Suppose $\phi(\bar{x})$ is the equation $t(\bar{x}) = 0$. Then, by Lemma 3.12, the set $\phi(\mathbb{Z}^n) \cap \Pi_q^n$ can be interpreted as a finite union of sets of the form

$$\bigwedge_{1 \leq i < n} x_i = s^{m_i}(x_{i+1}) \wedge \bigwedge_{1 \leq j \leq k} x_j \neq j.$$

Otherwise, suppose $\phi(\bar{x})$ is the congruence relation $s(\bar{x}) \equiv_l 0$. If (r_1, \dots, r_n) is a tuple of integers that satisfy the congruence relation, then any tuple (q_1, \dots, q_n) for $q_i \in [r_i]_l$ satisfies this relation. Note that we can only have finitely many solutions up to l -congruence. Moreover, we may assume, by the Chinese remainder theorem, that l is a power of a prime number. Thus, by Lemma 3.10 and Remark 3.11, $\phi(\mathbb{Z}^n) \cap \Pi_q^n$ can be interpreted as a finite union of sets of the form

$$\bigwedge_{1 \leq i \leq n} Q_{k_i, m_i}(x_i) \wedge "x_i \text{ is not equal to finitely many elements"}.$$

This finishes the proof. \square

Lemma 3.14. *The theory of $\mathcal{N} := (\mathbb{N}, s, \{Q_{k,n}\}_{n < \omega, k < n})$ admits quantifier elimination after adding a constant and a unary function symbol. Moreover it is superstable and has Lascar rank one.*

Proof. We add a constant to name 1 and a function symbol s^{-1} to name the predecessor function; observe that both are definable in \mathcal{N} .

We prove elimination of quantifiers by induction on the complexity of the formula ϕ . It is enough to consider the case where $\phi(\bar{x}, y)$ is a consistent formula of the form $\exists y \psi(\bar{x}, y)$, where $|y| = 1$ and $\psi(\bar{x}, y)$ is a quantifier free formula. We can clearly assume that ψ is in normal disjunctive form. Thus, since the negation of

$Q_{k,n}$ is equivalent to the conjunction $\bigvee_{l \neq k} Q_{l,n}$, it is enough to consider the case where $\psi(\bar{x}, y)$ is a finite conjunction of formulas of the following form:

$$Q_{k,n}(x_i) \wedge Q_{l,m}(y) \wedge x_i = c \wedge y = d \wedge x_i \neq a \wedge y \neq b \\ \wedge s^p(x_i) = x_j \wedge s^r(x_l) = y \wedge s^f(x_i) \neq x_j \wedge s^g(x_l) \neq y$$

Furthermore, we split ψ to a conjunction $\psi_0(\bar{x}, y) \wedge \psi_1(\bar{x})$, where ψ_1 is the conjunction of the atomic formulas of ψ that do not contain y . Clearly we may assume that $\psi_0(\bar{x}, y)$ does not contain instances of the form $y = d$ or $s^g(x_i) = y$. We claim that $\exists y \psi_0(\bar{x}, y)$ is equivalent to $\bar{x} = \bar{x}$. Indeed, the projection of any formula of the form

$$Q_{k,n}(y) \wedge \bigwedge_{1 \leq i \leq k} s^{g^i}(x) \neq y \wedge \bigwedge_{1 \leq j \leq l} y \neq d_j$$

is equivalent to $x = x$, thus the claim follows and $\psi(\bar{x}, y)$ is equivalent to $\psi_1(\bar{x})$. So, we obtain the first part of our statement.

Quantifier elimination allows us to prove by an easy counting types argument that the theory is superstable. Fix a set of parameters B . Clearly any non-algebraic type over B extends the set $\pi(x)$ given by $\{s^n(x) \neq a : a \in B, n \in \mathbb{Z}\}$. Whence, by the elimination of quantifiers, we obtain that any complete non-algebraic type over B (in one variable) is equivalent to $\pi(x) \cup \pi_0(x)$, where $\pi_0(x)$ is a complete type without parameters. Hence, $|S(B)| = |B| + |S(\emptyset)|$, as desired. In fact, any type without parameters is determined by positive formulas since, as noted before, the formula $\neg Q_{k,n}(x)$ is equivalent to a disjunction of formulas $Q_{l,n}(x)$ for $l \neq k$. In addition, as for any $n \in \mathbb{N}$ the formula $Q_{k,n}(x) \wedge Q_{l,n}(x)$ is inconsistent for distinct $l, k < n$, every complete type contains only one predicate of the form $Q_{k,n}(x)$ for a given n . Thus, it is easy to see that there are continuum many types without parameters; for instance, note that the predicate $Q_{k,2^n}(x)$ splits into $Q_{k,2^{n+1}}(x)$ and $Q_{k+2^n,2^{n+1}}(x)$ when k is odd. Hence $|S(B)| = |B| + 2^\omega$ and whence, the theory is not ω -stable.

Finally, again by quantifier elimination it is easy to see that the only formulas that divide are the algebraic ones. This shows that the theory has Lascar rank one; the details are left to the reader. \square

Now, the proof of Proposition 3.9 follows from Lemma 3.13 and 3.14. We can prove our second main theorem.

Proof of Theorem 2. It follows from Proposition 3.9 together with Theorem 3.2 that the expanded structure $(\mathbb{Z}, +, 0, \Pi_q)$ is superstable. As it is a proper expansion of $(\mathbb{Z}, +, 0)$, it has infinite Lascar rank by Theorem 1. Whence, it remains to see that it has Lascar rank ω . For this, it is enough to show that any forking extension of the principal generic has finite Lascar rank.

We shall work in an enough saturated extension of $(\mathbb{Z}, +, 0, \Pi_q)$, where Π_q is interpreted as Π'_q . Let $p \in S(\emptyset)$ be the generic of the connected component, and let $q = \text{tp}(b/B)$ be an extension of p . Consider a realization a of $p|B$, and note using Lemma 3.14 that Π'_q has Lascar rank one. Now, working in the theory of $(\mathbb{Z}, +, 0)$, we obtain that $\text{tp}(b/\Pi'_q, B)$ is the principal generic whenever $b \notin \text{acl}(\Pi'_q, B)$. Moreover, if a finite tuple d is algebraic over $\Pi'_q \cup B$ and this is exemplified by some finite tuple (c_1, \dots, c_n) in Π'_q , then we have in $\mathcal{Th}(\mathbb{Z}, +, 0, \Pi_q)$ that $U(d/B) \leq U(\bar{c}/B) < \omega$ as the set $\Pi'_q \times {}^n. \times \Pi'_q$ has Lascar rank n . Whence $a \notin \text{acl}(\Pi'_q, B)$ in the sense of

$(\mathbb{Z}, +, 0)$ and hence its type over $\Pi'_q \cup B$ is the principal generic. Thus, by Corollary 3.8 we deduce that $p|B = \text{tp}(b/B)$ whenever b is not algebraic in the sense of $(\mathbb{Z}, +, 0)$ over $\Pi'_q \cup B$. Therefore, in case that $\text{tp}(b/B)$ is a forking extension of p we conclude that $b \in \text{acl}(\Pi'_q, B)$ and so $\text{tp}(b/B)$ has finite Lascar rank, as desired. \square

One can see directly that the structure $(\mathbb{Z}, +, 0, \Pi_q)$ has infinite Lascar rank, without using Theorem 1, showing that the set $\Pi_q + \dots + \Pi_q$ has Lascar rank n . This is left to the reader.

4. GENERALIZATIONS

In this section we would like to mention a few generalizations, concerning proper superstable expansions of the integers, that follow from our methods. The ideas that lie behind our proof are transparent and clear. Firstly one reduces the superstability of the expanded structure to the superstability of the induced structure on the new predicate, for this a sufficient condition is that the new predicate defines a “sparse” set (see Remark 3.5). Secondly one needs to understand the induced structure in this new predicate. It seems that this is equivalent to understanding its intersection with arithmetic progressions and with the solution set of linear equations over the integers.

The following example is not very different in nature with the ones we already gave in the previous section, thus we leave its proof as an exercise to the interested reader.

Example 4.1. Let (k_1, \dots, k_m) be a sequence of natural numbers and

$$\text{SP}_{(k_1, \dots, k_m)} := \{k_1^{k_m^n} \mid n < \omega\}.$$

Then $(\mathbb{Z}, +, 0, \text{SP}_{(k_1, \dots, k_m)})$ is superstable of Lascar rank ω

A more interesting example is the subset of the integers consisting of factorial elements, i.e. $\text{Fac} := \{n! \mid n < \omega\} \cup \{0\}$.

Proposition 4.2. *The structure $(\mathbb{Z}, +, 0, \text{Fac})$ is superstable of Lascar rank ω .*

We first note that the set Fac satisfies the conditions of Remark 3.5, thus:

Lemma 4.3. *Let \mathcal{L} be the language of groups and $P(x)$ be a unary predicate. Let $\mathcal{Z} := (\mathbb{Z}, +, 0, \text{Fac})$ be an \mathcal{L}_P -structure. Then every \mathcal{L}_P -formula in \mathcal{Z} is bounded.*

We will next prove that the induced structure on Fac comes from equality alone.

Lemma 4.4. *Let $k < n$ be natural numbers and let $[k]_n$ denote the congruence class of k modulo n . Then $\text{Fac} \cap [k]_n$ is either finite or co-finite in Fac .*

Proof. It is easy to see that when k is 0 the intersection will be co-finite in Fac , while in any other case the intersection will be finite. \square

Given an equation $k_1x_1 + \dots + k_nx_n = 0$ over the integers and a partition $\mathcal{P} = \{I_j\}_{j \leq l}$ of $\{1, \dots, n\}$, we denote by $X_{\mathcal{P}}$ the set of solutions $(m_1!, \dots, m_n!)$ such that $m_i = m_k$ if and only if $i, k \in I_j$ for some $j \leq l$.

Lemma 4.5. *Let $k_1x_1 + \dots + k_nx_n = 0$ be an equation over the integers and let $\mathcal{P} = \{I_j\}_{j \leq l}$ be a partition of $\{1, \dots, n\}$. Then the projection of $X_{\mathcal{P}}$ on its I_j -coordinates is an infinite set if and only if $\sum_{i \in I_j} k_i = 0$.*

Proof. Let $\mathcal{P} = \{I_j\}_{j \leq l}$ be a partition of $\{1, \dots, n\}$ and suppose that $\sum_{i \in I_j} k_i = 0$ for some $j \leq l$. Clearly, there are infinitely many solution of the form (x_1, \dots, x_n) with $x_i = 0$ for $i \notin I_j$ and x_i constant for $i \in I_j$. Hence, we get the result. For the converse, assume for some $k \leq l$ that the projection of $X_{\mathcal{P}}$ on its I_k -coordinates yields an infinite set but $\sum_{i \in I_k} k_i$ is non-zero, and let $X_{\mathcal{P}}$ be the set $\{(m_1(t)!, \dots, m_n(t)!) \}_{t < \omega}$. Set $s_j(t)$ to be the value of every $m_i(t)$ when $i \in I_j$, and note that all $s_j(t)$'s are distinct by the definition of $X_{\mathcal{P}}$. It is clear that

$$\sum_{j \leq l} \left(\sum_{i \in I_j} k_i \right) \cdot s_j(t)! = 0.$$

Now, let J be the set of sub-indexes $j \leq l$ for which $\sum_{i \in I_j} k_i$ is non-zero; note that J is non-empty as $k \in J$ and also that

$$\sum_{j \in J} \left(\sum_{i \in I_j} k_i \right) \cdot s_j(t)! = 0.$$

By assumption, this equation holds for all $t < \omega$ and so, by the pigeonhole principle we can find an enumeration of $J = \{j_1, \dots, j_r\}$ such that $s_{j_1}(t) > \dots > s_{j_r}(t)$ for infinitely many values of t . Additionally, for some of these t 's we have that $s_{j_1}(t) > |\sum_{i \in I_{j_2}} k_i + \dots + \sum_{i \in I_{j_r}} k_i|$ and thus

$$\left| \left(\sum_{i \in I_{j_1}} k_i \right) \cdot s_{j_1}(t)! \right| > \left| \left(\sum_{i \in I_{j_2}} k_i \right) \cdot s_{j_2}(t)! + \dots + \left(\sum_{i \in I_{j_r}} k_i \right) \cdot s_{j_r}(t)! \right|,$$

a contradiction. Hence, we get the result. \square

If $k_1 x_1 + \dots + k_n x_n = 0$ is an equation over the integers and $S \subseteq \mathbb{Z}^n$ is its solution set, observe that S is precisely the finite union of all $X_{\mathcal{P}}$. Therefore, Lemmata 4.4 and 4.5 give that all the induced structure on Fac comes from equality alone, thus Fac^{ind} is strongly minimal and Proposition 4.2 follows.

Our paper can be seen as opening the path for answering the following interesting questions:

Question 4.6.

- (*J. Goodrick*) Characterize the subsets $\Pi \subset \mathbb{Z}$, for which $(\mathbb{Z}, +, 0, \Pi)$ is superstable.
- Characterize the subsets $\Pi \subset \mathbb{Z}$, for which $(\mathbb{Z}, +, 0, \Pi)$ is stable.

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